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Tensor Product of Cyclic Subgroup Graph of Finite Group

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Abstract

The cyclic subgroup graph $\Gamma_z(G)$ for a finite group G is a simple undirected graph in which the cyclic subgroups are vertices and two distinct subgroups are adjacent if one of them is a subset of the other. In this paper, we have discussed some properties and parameters of tensor product of cyclic subgroup graph of finite group in detail and obtained some theorems on it.

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1.Introduction

The study of algebraic structures using the properties of graphs has become an exciting research topic for the past three decades, leading to many fascinating results and results. Ma, X.L., H.Q. Wei and Guozhong(2013) have introduced the concept of Cyclic graph of a finite group. After that, J. John Arul Singh and S. Devi have introduced Cyclic Subgroup Graph of a finite group. The concept of split domination number was introduced by Kulli and Janakiram . Fink and Jacobson introduced the notion of k −domination in graphs. In this paper, we have discussed some properties and parameters of all these structures in detail and obtained some theorems on tensor product of cyclic subgroup graphs on finite groups.

2. Preliminaries

Definition:2.1[3] The *cyclic subgroup graph* $\Gamma_z(G)$ for a finite group G is a simple undirected graph in which the cyclic subgroups are vertices and two distinct subgroups are adjacent if one of them is a subset of the other.

Definition:2.2 [1] A graph H is *connected* if there is a path between every couple of vertices in H. A graph is said to be *disconnected* if it is not connected.

Definition:2.3 [1] A graph in which any two distinct points are adjacent is called a *complete* graph.

Definition: 2.4 [6] A graph is said to be *biconnected* if

(i) It is connected, i.e. it is possible to reach every vertex from every other vertex, by a simple path

(ii) Even after removing any vertex the graph remains connected.

Definition: 2.5 The maximum distance between a vertex to all other vertices is considered as the *eccentricity* of the vertex . The maximum eccentricity from all the vertices is considered as the *diameter* of the graph G. The minimum eccentricity from all the vertices is considered as the *radius* of the graph G.

Definition: 2.6 [1] For any graph G, the *maximum degree* is the largest number of neighbours of a vertex in G,i.e. $\Delta(G) = \max\{\deg v/v \in V(G)$ and *minimum degree* is the smallest number of neighbours of a vertex in G,i.e. $\delta(G) = \min\{\deg v/v \in V(G)$.

Definition: 2.7 [7] A *dominating set* for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one member of D. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for G.

Definition: 2.8 [7] **A** k-*dominating set* is a set of vertices D such that each vertex in $< V \setminus D >$ is dominated by at least k vertices in D for a fixed positive integer k. The minimum cardinality of a k-dominating set is called k-domination number $\gamma_k(G)$.

Definition: 2.9 [7] A dominating set D of a graph $G = (V, E)$ is a *split dominating set*, if the induced subgraph $< V \setminus D >$ is disconnected. The split domination number $\gamma_s(G)$ of G is the minimum cardinality of a split dominating set.

Definition: 2.10 [1] The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colours needed to colour G.

Definition:2.11 [5] The *tensor product* of graphs G and H is denoted by $G \times H$, whose vertex set is $V(G) \times V(H)$ and for which vertices (g, h) and (g', h') are adjacent precisely when $gg' \in E(G)$ and $hh' \in E(H)$.

Definition: 2.12 A *Clique* C, in an undirected graph $G = (V, E)$ is a subset of the vertices $C \subseteq V$, such that every two distinct vertices are adjacent. The induced subgraph induced by C is a complete graph.

Definition: 2.13 Let G be a graph of order n. Then $d_1, d_2, \ldots, d_n$ is a *degree sequence* of G if and only if the vertices of G can be labeled $v_1, v_2, \ldots, v_n$ such that $\deg(v_i) = d_i$ for all $i = 1,2, \ldots, n$.

Definition: 2.14 A graph $G = (V, E)$ is said to be *planar* if it can be drawn in the plane so that no two edges of G intersect at a point other than a vertex. Such a drawing of a planar graph is called planar embedding of a graph.

3. Main Results

Theorem: 3.1 The graph $G = \Gamma_z(\mathbb{Z}_{p^2}) \times \Gamma_z(\mathbb{Z}_p)$ for any prime p, is a cycle with six vertices. Then

(i) $\gamma_k(G) = \gamma_s(G) = \gamma(G)$

(ii) $\chi(G) = 2$

Proof: Let p be any prime number. By theorem [6], The cyclic subgroup graph $\Gamma_z(\mathbb{Z}_p)$ for any prime p, is a tree if and only if $\Gamma_z(\mathbb{Z}_p)$ is isomorphic to K_2 and the cyclic subgroup graph $\Gamma_z(\mathbb{Z}_{p^2})$ for any prime p, is isomorphic to K_3. Here the vertices from $\Gamma_z(\mathbb{Z}_{p^2})$ denoted as u, v, w and the vertices from $\Gamma_z(\mathbb{Z}_p)$ be u', v'. The obtained vertex set in the tensor product of cyclic subgroup graph, $V(\Gamma_z(\mathbb{Z}_{p^2}) \times \Gamma_z(\mathbb{Z}_p))$= $\{(u, u'), (u, v'), (v, u'), (v, v'), (w, u'), (w, v')\}$. By the definition of tensor product, (g, h) and (g', h') are adjacent in $G \times H$ if and only if g is adjacent to g' and h is adjacent to h'. Here, every vertex in the vertex set is adjacent to exactly two vertices. It is clear that, the resulting graph is a cycle with six vertices.

(i) Obviously, this is a cycle graph with six vertices. By selecting any 2 vertices which is not adjacent to each other makes a dominating set. The minimum cardinality of the dominating set is 2. Now D be the dominating set where each vertex in $V(G) - D$ is dominated by atleast 2 vertices in D for a fixed positive integer k. For this cycle graph, the induced subgraph will be disconnected when the dominating set is removed from the graph. The minimum cardinality of the split dominating set is 2.

Therefore, $\gamma_k(G) = \gamma_s(G) = \gamma(G)$

(ii) For a graph, no two adjacent vertices should have the same colour. The graph represented here is a cycle graph with six vertices. Therefore, assign colours for alternative vertices to paint the graph. The minimum colours used to colour the graph is 2. Therefore, $\chi(G) = 2$.

Theorem: 3.2 The graph $G = \Gamma_z(\mathbb{Z}_p) \times \Gamma_z(\mathbb{Z}_p)$ for any prime p, is disconnected.

Proof: Let p be any prime number. By theorem [6], The cyclic subgroup graph $\Gamma_z(\mathbb{Z}_p)$ for any prime p, is a tree if and only if $\Gamma_z(\mathbb{Z}_p)$ is isomorphic to K_2. Here the vertices from $\Gamma_z(\mathbb{Z}_p)$ denoted as u, v and u', v'. The obtained vertex set in the tensor product of cyclic subgroup graph , $V(\Gamma_z(\mathbb{Z}_p) \times \Gamma_z(\mathbb{Z}_p))$= $\{(u,u'),(u,v'),(v,u'),(v,v')\}$. Now the vertices are adjacent in $\Gamma_z(\mathbb{Z}_p) \times \Gamma_z(\mathbb{Z}_p)$ if u is adjacent to u' and v is adjacent to v'. In this graph, (u,u') is adjacent to (v,v') and (u,v') is adjacent to (v,u'). But (u,u') is not adjacent to (u,v')& (v,u'). Similarly, (v,v') is not adjacent to (u,v')&(v,u'). Hence, the resulting graph has two components both are isomorphic to complete graph K_2. It is clear that, the graph G is disconnected.

Theorem: 3.3 The graph $G = \Gamma_z(\mathbb{Z}_{p^2}) \times \Gamma_z(\mathbb{Z}_{p^2})$ for any prime p, is connected. Then

(i) $\gamma_k(G) = \gamma(G)$

(ii) $\gamma_s(G) = \gamma(G) + 2$

(iii) $\chi(G) = 4$

Proof: Let p be any prime number. By theorem [6], the cyclic subgroup graph $\Gamma_z\left(\mathbb{Z}_{p^2}\right)$ for any prime p, is isomorphic to K_3. The vertices from $\Gamma_z\left(\mathbb{Z}_{p^2}\right)$ be u, v, w and by taking product with the same group that denotes to be u', v', w'. The vertex set obtained in the tensor product of cyclic subgroup graph be $V(\Gamma_z(\mathbb{Z}_{p^2}) \times \Gamma_z(\mathbb{Z}_{p^2}))$=.

$\{(u,u'),(u,v'),(u,w'),(v,u'),(v,v'),(v,w'),(w,u'),(w,v'),(w,w')\}$

By the definition of tensor product, (g,h) and (g',h') are adjacent in $G \times H$ if and only if g is adjacent to g' and h is adjacent to h'. Now, every vertex belongs to the vertex set is adjacent to exactly 4 vertices, therefore the maximum degree and the minimum degree are same. It is clear that, there is a path in between every vertices which makes the graph connected.

(i) Clearly, the maximum degree is 4, choose any three vertices which is not adjacent to each other. The minimum cardinality of the dominating set is 3. Now D be the dominating set where each vertex in $V(G) - D$ is dominated by atleast 3 vertices in D for a fixed positive integer k. Therefore, $\gamma_k(G) = \gamma(G)$.

(ii) For this connected graph, the induced subgraph will be disconnected by removing 5 vertices. The minimum cardinality of the split dominating set is 5.
Therefore, $\gamma_s(G) = \gamma(G) + 2$

(iii) For a graph, no two adjacent vertices should have the same colour. The graph represented here is a connected graph. Therefore, assign colours for alternative vertices to paint the graph. The minimum colours used to colour the graph is 4.
Therefore, $\chi(G) = 4$.

Theorem: 3.4 The graph $G = \Gamma_z(\mathbb{Z}_{pq}) \times \Gamma_z(\mathbb{Z}_p)$ is connected where p & q be distinct primes then

(i) $\gamma_k(G) = \gamma(G) = \gamma_s(G)$

(ii) $\chi(G) = 3$

Proof: Let p *and* q be distinct primes where q should be greater than p. By theorem [6], the graph $\Gamma_z(G)$ for a finite group $G = \mathbb{Z}_{pq}$ where $q > p$ and p, q are distinct primes then $\Gamma_z(G)$ is isomorphic to $K_4 - e$ *and* The cyclic subgroup graph $\Gamma_z(\mathbb{Z}_p)$ for any prime p, is a tree if and only if $\Gamma_z(\mathbb{Z}_p)$ is isomorphic to K_2. The vertices from $\Gamma_z(\mathbb{Z}_{pq})$ be u, v, w, x and the vertices from $\Gamma_z(\mathbb{Z}_p)$ be u', v'. The vertex set obtained in the tensor product of cyclic subgroup graph be $V(\Gamma_z(\mathbb{Z}_{pq}) \times \Gamma_z(\mathbb{Z}_p))$=. $\{(u,u'), (u,v'), (v,u'), (v,v'), (w,u'), (w,v'), (x,u'), (x,v')\}$

By the definition of tensor product, (g, h) and (g', h') are adjacent in $G \times H$ if and only if g is adjacent to g' and h is adjacent to h'. In the graph $\Gamma_z(\mathbb{Z}_{pq})$, v and w are not adjacent to each other. Therefore, the vertices (v, v') (w, u') and (v, u') (w, v') are not adjacent to each other. Except these vertices all other vertices are neighbour to exactly three vertices therefore the maximum degree is twice the minimum degree. It is clear that, there is a path from a vertex to any other vertex which makes the graph connected.

(i) In this graph, two vertices is enough to dominate the remaining vertices. The minimum cardinality of the dominating set is 2. Now D be the dominating set where each vertex in $V(G) - D$ is dominated by atleast 2 vertices in D for a fixed positive integer k. Therefore, $\gamma_k(G) = \gamma(G)$.For this connected graph, the induced subgraph will be disconnected when the dominating set is removed from the graph.. The minimum cardinality of the split dominating set is 2. Therefore, $\gamma_s(G) = \gamma(G)$

(ii) For a graph, no two adjacent vertices should have the same colour. The graph represented here is a connected graph. Therefore, assign colours for alternative vertices to paint the graph. The minimum colours used to colour the graph is 3.Therefore, $\chi(G) = 3$.

Theorem: 3.5 The graph $\Gamma_z(G)$ is connected then the clique number is atmost 3.

Proof follows from the above theorem

Theorem: 3.6 The graph $G = \Gamma_z(\mathbb{Z}_{pq}) \times \Gamma_z\left(\mathbb{Z}_{p^2}\right)$ is biconnected where p & q be distinct primes then

(i) $\gamma_k(G) = \gamma(G)$

(ii) $\gamma_s(G) = \gamma(G) + 2$

(iii) $\chi(G) = 3$

Proof: Let p *and* q be distinct primes where q should be greater than p. By theorem [6], the graph $\Gamma_z(G)$ for a finite group $G = \mathbb{Z}_{pq}$ where $q > p$ and p, q are distinct primes then $\Gamma_z(G)$ is isomorphic to $K_4 - e$ *and* The cyclic subgroup graph $\Gamma_z\left(\mathbb{Z}_{p^2}\right)$ for any prime p, is isomorphic to K_3. The vertices from $\Gamma_z(\mathbb{Z}_{pq})$ be u, v, w, x and the vertices from $\Gamma_z\left(\mathbb{Z}_{p^2}\right)$ be u', v', w'. The vertex set obtained in the tensor product of cyclic subgroup graph be $V(\Gamma_z(\mathbb{Z}_{pq}) \times \Gamma_z\left(\mathbb{Z}_{p^2}\right))$=.

$\{(u,u'), (u,v'), (u,w'), (v,u'), (v,v'), (v,w')\ (w,u'), (w,v'), (w,w')(x,u'), (x,v'), (x,w')\}$

By the definition of tensor product, (g,h) and (g',h') are adjacent in $G \times H$ if and only if g is adjacent to g' and h is adjacent to h'. In the graph $\Gamma_z(\mathbb{Z}_{pq})$, v and w are not adjacent to each other. Therefore, the vertices $(v,u')\ (v,v'), (v,w')$ are not adjacent to $(w,u')\ (w,v'), (w,w')$. Excluding these vertices all other vertices are neighbour to exactly six vertices each. Thus $\Delta(G) = \delta(G) + 2$. It is clear that, there is a path between every starting vertex and ending vertex . Evenafter removing any vertex the graph remains connected. Therefore, the graph G does not have an articulation point. Now, concluding that G is connected and it does not contain any articulation point, which results to a biconnected graph.

(iv) In this graph, two vertices is enough to dominate the remaining vertices. The minimum cardinality of the dominating set is 3. Now D be the dominating set where each vertex in $V(G) - D$ is dominated by atleast 3 vertices in D for a fixed positive integer k. Therefore, $\gamma_k(G) = \gamma(G)$.

(v) For this connected graph, the induced subgraph will be disconnected by removing 5 vertices . The minimum cardinality of the split dominating set is 5.
Therefore, $\gamma_s(G) = \gamma(G) + 2$

(vi) For a graph, no two adjacent vertices should have the same colour. The graph represented here is a connected graph. Therefore, assign colours for alternative vertices to paint the graph. The minimum colours used to colour the graph is 3.
Therefore, $\chi(G) = 3$.

Theorem: 3.7 The graph $G = \Gamma_z(\mathbb{Z}_{pq}) \times \Gamma_z(\mathbb{Z}_{pq})$ is biconnected where p & q be distinct primes then

(i) $\gamma_k(G) = \gamma(G)$
(ii) $\gamma_s(G) = \gamma(G) + 3$
(iii) $\chi(G) = 3$

Proof is obvious from the above theorem.

We have discussed some properties regarding tensor product of cyclic subgroup graphs in the following table.

Graph G	Radius	Diameter	Degree	Degree Sequence	Planarity
$\Gamma_z(\mathbb{Z}_{p^2}) \times \Gamma_z(\mathbb{Z}_p)$	3	3	$\Delta(G) = \delta(G)$	2,2,2,2,2	Planar
$\Gamma_z(\mathbb{Z}_{p^2}) \times \Gamma_z(\mathbb{Z}_{p^2})$	2	2	$\Delta(G) = \delta(G)$	4,4,4,4,4,4,4,4,4	Non-Planar
$\Gamma_z(\mathbb{Z}_{pq}) \times \Gamma_z(\mathbb{Z}_p)$	3	3	$\Delta(G) = \delta(G) + 1$	3,3,3,3,2,2,2,2	Non-Planar

$\Gamma_z(\mathbb{Z}_{pq}) \times \Gamma_z(\mathbb{Z}_{p^2})$	2	2	$\Delta(G) = \delta(G) + 2$	6,6,6,6,6,6,4,4,4,4	Non-Planar
$\Gamma_z(\mathbb{Z}_{pq}) \times \Gamma_z(\mathbb{Z}_{pq})$	2	2	$\Delta(G) = \delta(G) + 5$	9,9,9,9,6,6,6,6,6,6,4,4,4,4	Non-Planar

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